

# THE THEORY OF TRACIAL VON NEUMANN ALGEBRAS DOES NOT HAVE A MODEL COMPANION

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**ABSTRACT.** In this note, we show that the theory of tracial von Neumann algebras does not have a model companion. This will follow from the fact that the theory of any locally universal, McDuff  $\text{II}_1$  factor does not have quantifier elimination. We also show how a positive solution to the Connes Embedding Problem implies that there can be no model-complete theory of  $\text{II}_1$  factors.

## 1. INTRODUCTION

The model theoretic study of operator algebras is at a relatively young stage in its development (although many interesting results have already been proven, see [7],[8], [9]) and thus there are many foundational questions that need to be answered. In this note, we study the question that appears in the title: does the theory of tracial von Neumann algebras have a model companion? (Recall that a theory is said to be *model-complete* if every embedding between models of the theory is elementary and a model-complete theory  $T'$  is a *model companion* of a theory  $T$  if every model of  $T$  embeds into a model of  $T'$  and vice-versa.) We show that the answer to this question is: no! Indeed, we prove that a locally universal, McDuff  $\text{II}_1$  factor cannot have quantifier elimination. (See below for the definitions of *locally universal* and *McDuff*.) Since a model companion of the theory of tracial von Neumann algebras will have to be a model completion as well as the theory of a locally universal, McDuff  $\text{II}_1$  factor, the result follows.

We then pose a weaker question: can there exist a model-complete theory of  $\text{II}_1$  factors? Here, we show that a positive solution to the *Connes Embedding Problem* implies that the answer is once again: no!

Another motivation for this work came from considering independence relations in  $\text{II}_1$  factors. Although all  $\text{II}_1$  factors are unstable (see [7]), it is still possible that there are other reasonably well-behaved independence relations to consider. Indeed, the independence relation stemming from conditional expectation is a natural candidate. In the end of this note, we show how the failure of quantifier elimination seems to pose serious hurdles in showing that conditional expectation yields a strict independence relation in the sense of [1].

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We thank Dima Shlyakhtenko for patiently explaining Brown's work when we posed the question to him of the existence of non-extendable embeddings of pairs  $\mathcal{M} \subset \mathcal{N}$  into  $\mathcal{R}^\omega$ . (See the proof of Theorem 2.1 below.)

Throughout,  $\mathcal{L}$  denotes the signature for tracial von Neumann algebras and  $\mathcal{R}$  denotes the hyperfinite  $\text{II}_1$  factor. We recall that  $\mathcal{R}$  embeds into any  $\text{II}_1$  factor. We will say that a von Neumann algebra is  $\mathcal{R}^\omega$ -embeddable if it embeds into  $\mathcal{R}^\mathcal{U}$  for some  $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ . If  $M$  is  $\mathcal{R}^\omega$  embeddable, then  $M$  embeds into  $\mathcal{R}^\mathcal{U}$  for all  $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ ; see Corollary 4.15 of [8]. For this reason, we fix  $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$  throughout this note.

## 2. MODEL COMPANIONS

In the proof of our first theorem, we use the crossed product construction for von Neumann algebras; a good reference is [4, Chapter 4].

**Theorem 2.1.**  *$\text{Th}(\mathcal{R})$  does not have quantifier elimination.*

*Proof.* It is enough to find separable,  $\mathcal{R}^\omega$ -embeddable tracial von Neumann algebras  $M \subset N$  and an embedding  $\pi : M \rightarrow \mathcal{R}^\mathcal{U}$  that does not extend to an embedding  $N \rightarrow \mathcal{R}^\mathcal{U}$ . Indeed, if this is so, let  $N_1$  be a separable model of  $\text{Th}(\mathcal{R})$  containing  $N$ . Then  $\pi$  does not extend to an embedding  $N_1 \rightarrow \mathcal{R}^\mathcal{U}$ ; since  $\mathcal{R}^\mathcal{U}$  is  $\aleph_1$ -saturated, this shows that  $\text{Th}(\mathcal{R})$  does not have QE.

In order to achieve the goal of the above paragraph, we claim that it is enough to find a countable discrete group  $\Gamma$  such that  $L(\Gamma)$  is  $\mathcal{R}^\omega$ -embeddable, an embedding  $\pi : L(\Gamma) \rightarrow \mathcal{R}^\mathcal{U}$ , and  $\alpha \in \text{Aut}(L(\Gamma))$  such that there exists no unitary  $u \in \mathcal{R}^\mathcal{U}$  satisfying  $(\pi \circ \alpha)(x) = u\pi(x)u^*$  for all  $x \in L(\Gamma)$ . (We should remark that we are using the usual trace on  $L(\Gamma)$  and that  $\text{Aut}(L(\Gamma))$  refers to the group of  $*$ -automorphisms preserving this trace.) First, we abuse notation and also use  $\alpha$  to denote the homomorphism  $\mathbb{Z} \rightarrow \text{Aut}(L(\Gamma))$  which sends the generator of  $\mathbb{Z}$  to the aforementioned  $\alpha$ . Set  $\mathcal{M} = L(\Gamma)$  and  $\mathcal{N} = \mathcal{M} \rtimes_\alpha \mathbb{Z}$ . Then  $N$  is a tracial von Neumann algebra. Moreover, we have that  $\mathcal{N}$  is  $\mathcal{R}^\omega$ -embeddable if and only if  $\mathcal{M}$  is—in fact, this is true for any crossed product algebra  $\mathcal{M} \rtimes_\alpha G$  where  $G$  is amenable [2, Prop. 3.4(2)]. Now suppose, towards a contradiction, that  $\pi$  were to extend to an embedding  $\tilde{\pi} : \mathcal{N} \rightarrow \mathcal{R}^\mathcal{U}$ . If  $u \in L(\mathbb{Z}) \subset \mathcal{M} \rtimes_\alpha \mathbb{Z}$  is the generator of  $\mathbb{Z}$ , then setting  $\tilde{u} = \tilde{\pi}(u) \in \mathcal{R}^\mathcal{U}$ , we would have that  $\tilde{u}\pi(x)\tilde{u}^* = \pi(uxu^*) = \pi(\alpha(x))$  for all  $x \in \mathcal{M}$ , contradicting the fact that  $\pi \circ \alpha$  is not unitarily conjugate to the embedding  $\pi$  in  $\mathcal{R}^\mathcal{U}$ .

An explicit construction of  $\Gamma$ ,  $\pi$  and  $\alpha$  as above has already appeared in the work of N. P. Brown [6]. Indeed, by Corollary 6.11 of [6], we may choose  $\Gamma = \text{SL}(3, \mathbb{Z}) * \mathbb{Z}$  and  $\alpha = \text{id} * \theta$  for any nontrivial  $\theta \in \text{Aut}(L(\mathbb{Z}))$ . □

We say that a separable  $\text{II}_1$  factor  $\mathcal{S}$  is *locally universal* if every separable  $\text{II}_1$  factor embeds into  $\mathcal{S}^\mathcal{U}$ . (By [8, Corollary 4.15], this notion is independent of  $\mathcal{U}$ .) In [9], it is shown that a locally universal  $\text{II}_1$  factor exists. The *Connes Embedding Problems* (CEP) asks whether  $\mathcal{R}$  is locally universal.

We say that a separable  $\text{II}_1$  factor  $M$  is *McDuff* if  $M \otimes \mathcal{R} \cong M$ . For example,  $\mathcal{R}$  is McDuff as is  $M \otimes \mathcal{R}$  for any separable  $\text{II}_1$  factor  $M$ . By examining Brown's argument in [6], we see that the only properties of  $\mathcal{R}$  that are used (other than it being finite) is that  $L(\Gamma)$  (for  $\Gamma$  as in the previous proof) is  $\mathcal{R}^\omega$ -embeddable and that  $\mathcal{R}$  is McDuff. We thus have:

**Corollary 2.2.** *If  $\mathcal{S}$  is a locally universal, McDuff  $\text{II}_1$  factor, then  $\text{Th}(\mathcal{S})$  does not have QE.*

Let  $T_0$  be the theory of tracial von Neumann algebras in the signature  $\mathcal{L}$ .  $T_0$  is a universal theory; see [8]. Let  $T$  be the theory of  $\text{II}_1$  factors, a  $\forall\exists$ -theory by [8]. Moreover, since every tracial von Neumann algebra is contained in a  $\text{II}_1$  factor, we see that  $T_0 = T_\forall$ . Thus, an existentially closed model of  $T_0$  is a model of  $T$ .

By [9, Proposition 3.9], there is a set  $\Sigma$  of  $\forall\exists$ -sentences in the language of tracial von Neumann algebras such that  $M$  is McDuff if and only if  $M \models \Sigma$ . Since every  $\text{II}_1$  factor is contained in a McDuff  $\text{II}_1$  factor (as  $M \subseteq M \otimes \mathcal{R}$ ), it follows that an existentially closed  $\text{II}_1$  factor is McDuff.

We can now prove our main result:

**Theorem 2.3.**  *$T_0$  does not have a model companion.*

*Proof.* Suppose that  $T$  is a model companion for  $T_0$ . Since  $T_0$  is universally axiomatizable and has the amalgamation property (see [4, Chapter 4]), we have that  $T$  has QE.

Fix a separable model  $\mathcal{S}$  of  $T$ . Then  $\mathcal{S}$  is a locally universal  $\text{II}_1$  factor. Indeed, given an arbitrary separable  $\text{II}_1$  factor  $M$ , we have a separable model  $\mathcal{S}_1 \models T$  containing  $M$ . Since  $\mathcal{S}^\omega$  is  $\aleph_1$ -saturated, we have that  $\mathcal{S}_1$  embeds into  $\mathcal{S}^\omega$ , yielding an embedding of  $M$  into  $\mathcal{S}^\omega$ . Meanwhile, since  $T$  is the theory of existentially closed models of  $T_0$ , we see that  $\mathcal{S}$  is McDuff. Thus, by Corollary 2.2,  $T$  does not have QE, a contradiction.  $\square$

### 3. MODEL COMPLETE $\text{II}_1$ FACTORS

While we have proven that the theory of tracial von Neumann algebras does not have a model companion, at this point it is still possible that there is a model complete theory of  $\text{II}_1$  factors. In this section, we show that a positive solution to the CEP implies that there is no model-complete theory of  $\text{II}_1$  factors.

We begin by observing the following:

**Lemma 3.1.** *Every embedding  $\mathcal{R} \rightarrow \mathcal{R}^\omega$  is elementary.*

*Proof.* This follows from the fact that every embedding  $\mathcal{R} \rightarrow \mathcal{R}^\omega$  is unitarily equivalent to the diagonal embedding; see [10].  $\square$

*Remark.* The previous lemma shows that  $\mathcal{R}$  is the unique prime model of its theory. Indeed, to show that  $\mathcal{R}$  is a prime model of its theory, by Downward

Löwenheim-Skolem (DLS), it is enough to show that whenever  $M \equiv \mathcal{R}$  is separable, then  $\mathcal{R}$  elementarily embeds into  $M$ . Well, since  $\mathcal{R}^{\mathcal{U}}$  is  $\aleph_1$ -saturated, we have that  $M$  elementarily embeds into  $\mathcal{R}^{\mathcal{U}}$ . Composing an embedding  $\mathcal{R} \rightarrow M$  with the elementary embedding  $M \rightarrow \mathcal{R}^{\mathcal{U}}$  and applying Lemma 3.1, we see that the embedding  $\mathcal{R} \rightarrow M$  is elementary.

**Proposition 3.2.** *Suppose that  $M$  is an  $\mathcal{R}^{\omega}$ -embeddable  $II_1$  factor such that  $\text{Th}(M)$  is model-complete. Then  $M \equiv \mathcal{R}$ .*

*Proof.* Without loss of generality, we may assume that  $M$  is separable. Fix embeddings  $\mathcal{R} \rightarrow M$  and  $M \rightarrow \mathcal{R}^{\mathcal{U}}$ . By Lemma 3.1, the composition

$$\mathcal{R} \rightarrow M \rightarrow \mathcal{R}^{\mathcal{U}}$$

is elementary. By DLS, we can take a separable elementary substructure  $\mathcal{R}_1$  of  $\mathcal{R}^{\mathcal{U}}$  such that  $M$  embeds in  $\mathcal{R}_1$ ; observe that the composition  $\mathcal{R} \rightarrow M \rightarrow \mathcal{R}_1$  is elementary. By DLS again, take a separable elementary substructure  $M_1$  of  $M^{\mathcal{U}}$  such that  $\mathcal{R}_1$  embeds in  $M_1$ . We now repeat this process with  $M_1$ : embed  $M_1$  in  $\mathcal{R}^{\mathcal{U}}$ , take separable elementary substructure  $\mathcal{R}_2$  of  $\mathcal{R}^{\mathcal{U}}$  such that  $M_1$  embeds in  $\mathcal{R}_2$  and then embed  $\mathcal{R}_2$  in a separable elementary substructure  $M_2$  of  $M^{\mathcal{U}}$ . Iterate this construction countably many times, obtaining

$$\mathcal{R} \rightarrow M \rightarrow \mathcal{R}_1 \rightarrow M_1 \rightarrow \mathcal{R}_2 \rightarrow M_2 \rightarrow \cdots,$$

where each  $\mathcal{R}_n$  is a separable elementary substructure of  $\mathcal{R}^{\mathcal{U}}$  and each  $M_i$  is a separable elementary substructure of  $M^{\mathcal{U}}$ . Set  $\mathcal{R}_{\omega} = \bigcup_n \mathcal{R}_n = \bigcup_n M_n$ . Then  $\mathcal{R}$  is an elementary substructure of  $\mathcal{R}_{\omega}$  since  $\mathcal{R} \rightarrow \mathcal{R}_1$  is elementary and  $\mathcal{R}_n \rightarrow \mathcal{R}_{n+1}$  is elementary for each  $n \geq 1$ . Meanwhile, observe that  $M_n \equiv M$  for each  $n$ , so by model-completeness of  $\text{Th}(M)$ , we have that the  $M_n$ 's form an elementary chain, whence  $M$  is an elementary substructure of  $\mathcal{R}_{\omega}$ . Consequently,  $\mathcal{R} \equiv M$ .  $\square$

**Remark 3.3.** Proposition 3.2 provides immediate examples of non-model complete theories of  $II_1$  factors. Indeed, for  $m \geq 2$ , the von Neumann group algebra of the free group on  $m$  generators,  $L(\mathbb{F}_m)$ , is  $\mathcal{R}^{\omega}$ -embeddable but not elementarily equivalent to  $\mathcal{R}$  (see 3.2.2 in [9]), whence  $\text{Th}(L(\mathbb{F}_m))$  is not model-complete. It is an outstanding problem in operator algebras whether or not  $L(\mathbb{F}_m) \cong L(\mathbb{F}_n)$  for all  $m, n \geq 2$ . A weaker, but still seemingly difficult, question is whether or not  $L(\mathbb{F}_m) \equiv L(\mathbb{F}_n)$  for all  $m, n \geq 2$ . (An equivalent formulation of this question is whether or not there is  $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$  such that  $L(\mathbb{F}_m)^{\mathcal{U}} \cong L(\mathbb{F}_n)^{\mathcal{U}}$ ?) Suppose this latter question has an affirmative answer. Then we see that the theory of free group von Neumann algebras is not model-complete, mirroring the corresponding fact that the theory of free groups is not model-complete. However, the natural embeddings  $\mathbb{F}_m \rightarrow \mathbb{F}_n$ , for  $m < n$ , are elementary. Assuming  $L(\mathbb{F}_m) \equiv L(\mathbb{F}_n)$ , are the natural embeddings  $L(\mathbb{F}_m) \rightarrow L(\mathbb{F}_n)$ , for  $m < n$ , elementary?

**Corollary 3.4.** *Assume that the CEP has a positive solution. Then there is no model-complete theory of  $II_1$  factors.*

*Proof.* Suppose that  $T$  is a model-complete theory of  $\text{II}_1$  factors. By the positive solution to the CEP and Proposition 3.2,  $T = \text{Th}(\mathcal{R})$ . Meanwhile, a positive solution to the CEP implies that  $T_{\forall} = T_0$ , whence  $T$  is a model companion for  $T_0$ , contradicting Theorem 2.3.  $\square$

#### 4. CONCLUDING REMARKS

Theorem 2.1 presents a major hurdle in trying to understand the model theory of  $\text{II}_1$  factors. In particular, it places a major roadblock in trying to understand potential independence relations in theories of  $\text{II}_1$  factors. Indeed, although any  $\text{II}_1$  factor is unstable (see [7]), one might wonder whether the natural notion of independence stemming from noncommutative probability theory might show that some  $\text{II}_1$  factor is (real) rosy (see [1] for the definition of rosy theory). More precisely, fix some “large”  $\text{II}_1$  factor  $M$  and consider the relation  $\perp$  on “small” subsets of  $M$  given by  $A \perp_C B$  if and only if, for all  $a \in \langle AC \rangle$ ,  $E_{\langle C \rangle}(a) = E_{\langle BC \rangle}(a)$ . Here,  $\langle * \rangle$  denotes the von Neumann subalgebra generated by  $*$  and  $E_{\langle * \rangle}$  is the conditional expectation (or orthogonal projection) map  $E_{\langle * \rangle} : L^2 M \rightarrow L^2 \langle * \rangle$ . In trying to verify some of the natural axioms for an independence relation (see [1]), one runs into trouble when trying to verify the extension axiom: If  $B \subseteq C \subseteq D$  and  $A \perp_B C$ , can we find  $A'$  realizing the same type as  $A$  over  $C$  such that  $A' \perp_B D$ ? If  $M = \mathcal{R}^{\mathcal{U}}$  and “small” means “countable,” then it seems quite likely that one could find an  $A'$  with the same *quantifier-free type* as  $A$  over  $C$  that is independent from  $D$  over  $B$  as quantifier-free types are determined by moments. Without quantifier-elimination, it seems quite difficult to prove the extension property for this purported notion of independence. (The question of whether or not the independence relation arising from conditional expectation yields a strict independence relation was also discussed in [5].)

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